

# Using Nonlinear RF Acceleration for FEL Beam Conditioning

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## Abstract

We consider a new approach to condition an electron beam using nonlinear effects in the RF field. We demonstrate that such effects can generate a desirable—for the FEL interaction—radial variation of the particle's energy in the beam, and calculate the induced energy spread in the limit of weak field.

## INTRODUCTION

The most demanding requirement for future X-ray FELs [1, 2] is the generation of a sufficiently small transverse electron emittance. To mitigate this problem, an idea has been proposed in the past to “condition” an electron beam prior to the undulator of a Free-Electron Laser (FEL) by increasing each particle's energy in proportion to the square of its transverse betatron amplitude [3].

Several methods of beam conditioning were considered in the literature. The original idea [3] utilizes RF structures with a  $TM_{210}$  accelerating mode in a FODO lattice. Such a system turns out to be too long for parameters of modern x-ray FELs. In Ref. [4] (see also [5]), it was shown that a beam can be conditioned using solenoid magnets in combination with accelerating structures that generate energy chirp in the beam. Although such conditioner can be relatively short, unfortunately, it introduces a large projected emittance growth in the beam due to head-tail focusing variation during conditioning. In Ref. [6], it was shown that the emittance growth can be avoided in a specially designed lattices that satisfy matching conditions and use a more gentle pace of conditioning. Techniques considered in that paper are more appropriate for a specialized conditioning ring. A shorter device based on a “quadrupole undulator” was proposed in [7] which, however, imposes extremely tight tolerances for the magnetic field.

More recently, two conditioning methods were proposed that use lasers. In Ref. [8], conditioning is achieved via Thomson scattering of laser photons on the beam, and in Ref. [9] the beam is conditioned via modulation of its energy due to the interaction with the laser light in an undulator.

In this paper we consider a new mechanism of beam conditioning. It is based on an interaction of the beam with electromagnetic field in an axisymmetric RF structure and uses the effect of nonlinear acceleration (proportional to the square of the field). We show that such interaction introduces a parabolic energy profile in the beam which scales with the electric field  $E$  and the with the beam energy  $mc^2\gamma$  as  $\propto E^2\gamma^{-1}$ . The advantage of this approach is the direct modulation of the beam energy on a relatively short distance.

## FIELDS IN ACCELERATING CAVITY AND EQUATIONS OF MOTION

We assume time dependence of the electromagnetic field in an RF cavity  $\propto e^{-i\omega t}$  and use a paraxial approximation near the axis of the cavity. In this approximation, the field components are expressed in terms of the longitudinal electric field on the axis  $E_{z0}(z)$ :

$$\begin{aligned} E_z(r, z, t) &= \left[ E_{z0}(z) - \frac{r^2}{4} \left( \frac{\partial^2 E_{z0}}{\partial z^2} + \frac{\omega^2}{c^2} E_{z0} \right) \right] e^{-i\omega t} \\ &\quad + \text{c.c.}, \\ E_r(r, z, t) &= -\frac{r}{2} \frac{\partial E_{z0}}{\partial z} e^{-i\omega t} + \text{c.c.}, \\ B_\theta(r, z, t) &= -i \frac{r\omega}{2c} E_{z0} e^{-i\omega t} + \text{c.c.}, \end{aligned} \quad (1)$$

where c.c. stands for the complex conjugate, and we use the Gaussian system of units. We will use the Fourier representation for the function  $E_{z0}(z)$

$$E_{z0}(z) = \int_{-\infty}^{\infty} \mathcal{E}(\kappa) e^{i\kappa z} d\kappa, \quad (2)$$

then

$$\begin{aligned} E_z(r, z, t) &= \int_{-\infty}^{\infty} d\kappa e^{i\kappa z - i\omega t} \mathcal{E}(\kappa) \\ &\quad \times \left[ 1 - \frac{r^2}{4} \left( -\kappa^2 + \frac{\omega^2}{c^2} \right) \right] + \text{c.c.}, \\ E_r(r, z, t) &= -\frac{ir}{2} \int_{-\infty}^{\infty} d\kappa e^{i\kappa z - i\omega t} \mathcal{E}(\kappa) \kappa + \text{c.c.}, \\ B_\theta(r, z, t) &= -i \frac{r\omega}{2c} \int_{-\infty}^{\infty} d\kappa e^{i\kappa z - i\omega t} \mathcal{E}(\kappa) + \text{c.c.} \end{aligned} \quad (3)$$

The equation of motion for an electron passing through the RF cavity can be written in the following form

$$m\gamma \frac{d^2 \mathbf{R}}{dt^2} = e\mathbf{E} + \frac{e}{c} \mathbf{v} \times \mathbf{B} - \frac{e}{c^2} \mathbf{v}(\mathbf{v} \cdot \mathbf{E}), \quad (4)$$

where  $\mathbf{R}(t)$  is the electron radius-vector and  $\mathbf{v}(t)$  is the velocity. This equation is supplemented by the following equation for the relativistic factor  $\gamma$ :

$$mc^2 \frac{d\gamma}{dt} = e\mathbf{v} \cdot \mathbf{E}. \quad (5)$$

## ENERGY GAIN

We will solve the above equations using a perturbation approach in which the electric and magnetic fields are considered to be small, or first order. This also means the

expected changes in particle's momentum (and hence energy) and position are small. In our analysis we assume relativistic motion with  $\gamma \gg 1$ . The functions  $\gamma(t)$ ,  $\mathbf{R}(t)$ , and  $\mathbf{v}(t)$  are expanded into the Taylor series,

$$\begin{aligned}\gamma(t) &= \gamma_0(t) + \gamma_1(t) + \gamma_2(t) + \dots \\ \mathbf{R}(t) &= \mathbf{R}_0(t) + \mathbf{R}_1(t) + \dots \\ \mathbf{v}(t) &= \mathbf{v}_0(t) + \mathbf{v}_1(t) + \dots,\end{aligned}\quad (6)$$

where the subscript indicates the order of smallness of the quantity. The zeroth order quantities correspond to the motion with a constant velocity and neglect the effect of the fields in the cavity,

$$\mathbf{v}_0(t) = \beta c \hat{z}, \quad \mathbf{R}_0(t) = (\beta c t + z_0) \hat{z} + \mathbf{r}_0, \quad (7)$$

where  $\hat{z}$  is the unit vector in the direction of  $z$  axis,  $\mathbf{r}_0$  is a two dimensional vector of particle's offset perpendicular to  $\hat{z}$ , and  $z_0$  is the initial  $z$  coordinate of the particle at  $t = 0$ . Eqs. (7) represent the particle's unperturbed orbit parallel to the axis of the cavity.

Substituting Eqs. (7) into the right hand side of Eq. (4) gives an equation for  $\mathbf{R}_1(t)$ . Taking the radial component of this equation we obtain

$$\begin{aligned}m\gamma \frac{d^2 R_1}{dt^2} &= eE_r - e\beta B_\theta \\ &= -\frac{eir_0}{2} \int_{-\infty}^{\infty} d\kappa e^{i(\kappa\beta c - \omega)t} \hat{\mathcal{E}}(\kappa, z_0) \left( \kappa - \beta \frac{\omega}{c} \right) + \text{c.c.},\end{aligned}\quad (8)$$

where we use the notation

$$\hat{\mathcal{E}}(\kappa, z_0) = \mathcal{E}(\kappa) e^{i\kappa z_0}. \quad (9)$$

Integrating this equation we find the first order corrections to the particle's orbit and the transverse velocity,

$$\begin{aligned}R_1(t) &= \frac{eir_0}{2mc\gamma} \int_{-\infty}^{\infty} d\kappa e^{i(\kappa\beta c - \omega)t} \hat{\mathcal{E}}(\kappa, z_0) \frac{\kappa c - \beta\omega}{(\kappa\beta c - \omega)^2} \\ &\quad + \text{c.c.}, \\ v_{r1}(t) &= -\frac{er_0}{2mc\gamma} \int_{-\infty}^{\infty} d\kappa e^{i(\kappa\beta c - \omega)t} \hat{\mathcal{E}}(\kappa, z_0) \frac{\kappa c - \beta\omega}{\kappa\beta c - \omega} \\ &\quad + \text{c.c.}.\end{aligned}\quad (10)$$

The poles in the denominator of these equations should be treated as if the variable  $\kappa$  has an infinitely small negative imaginary part,  $\kappa \rightarrow \kappa - i\epsilon$ , with  $\epsilon > 0$  and  $\epsilon \rightarrow 0$ .

The first order equation for  $\gamma_1$  reads

$$mc^2 \frac{d\gamma_1}{dt} = eE_z \beta c. \quad (11)$$

Integrating it over time we can find the energy gain  $mc^2 \Delta\gamma_1 = mc^2 \gamma_1(t = \infty)$  after the passage through the cavity:

$$\begin{aligned}mc^2 \Delta\gamma_1 &= e\beta c \int_{-\infty}^{\infty} dt E_z \\ &= 2\pi e \left( \hat{\mathcal{E}}(\kappa_0, z_0) \left[ 1 - \frac{r_0^2}{4} \left( -\kappa_0^2 + \frac{\omega^2}{c^2} \right) \right] + \text{c.c.} \right) \\ &\approx 2\pi e \left( 1 + \frac{r_0^2 \omega^2}{4c^2 \gamma^2} \right) \left[ \hat{\mathcal{E}}(\kappa_0, z_0) + \text{c.c.} \right],\end{aligned}\quad (12)$$

where  $k_0 = \omega/\beta c$ , and we used approximation of large  $\gamma$ . The last equation demonstrates a well known fact that the linear acceleration in a cavity is due to the synchronous harmonic of the electric field with a phase velocity equal to the velocity of the particle ( $\omega/\kappa_0 = \beta c$ ). It also shows that the energy gain in linear acceleration has a weak radial dependence  $\propto r_0^2$ , however, for relativistic particles with  $\gamma \gg 1$  the effect is negligibly small, of the order of  $(kr_0/2\gamma)^2$ .

Using Eq. (10) one can also calculate the transverse velocity change of the particle  $\Delta v_1 = v_{r1}(t = \infty)$ . The result is

$$\begin{aligned}\Delta v_1 &= -\frac{i\pi e r_0}{mc\gamma\beta} \left( \kappa_0 - \beta \frac{\omega}{c} \right) \hat{\mathcal{E}}(\kappa_0, z_0) + \text{c.c.} \\ &\approx -\frac{i\pi e r_0 \omega}{mc^2 \gamma^3} \hat{\mathcal{E}}(\kappa_0, z_0) + \text{c.c.}\end{aligned}\quad (13)$$

Using Eqs. (12) and (13) and noting that  $\partial \hat{\mathcal{E}}(\kappa_0, z_0)/\partial z_0 = i\kappa_0 \hat{\mathcal{E}}(\kappa_0, z_0)$  it is easy to check that

$$c \frac{\partial \Delta\gamma_1}{\partial r_0} = \gamma \frac{\partial \Delta v_1}{\partial z_0}, \quad (14)$$

which relates the energy gain with the transverse kick. This formula is a manifestation of the Panofsky-Wentzel relation [10] usually invoked in the theory of wakefields.

## SECOND ORDER EFFECTS

We will now look into the second order effects in the energy gain. The equation for the second order  $d\gamma_2/dt$  reads

$$mc^2 \frac{d\gamma_2}{dt} = eE_r v_{r1} + e\beta c \frac{\partial E_z}{\partial r} r_1. \quad (15)$$

Substituting for  $v_{r1}$  and  $r_1$  from Eq. (10) and integrating over time from  $-\infty$  to  $\infty$  we find  $\Delta\gamma_2$ :

$$\begin{aligned}mc^2 \Delta\gamma_2 &= e \int_{-\infty}^{\infty} dt \left( E_r v_{r1} + \beta c \frac{\partial E_z}{\partial r} r_1 \right) \\ &= \frac{ir_0^2 e^2}{4mc\gamma} \left\{ \int_{-\infty}^{\infty} dt \left( \int d\kappa e^{i(\kappa\beta c - \omega)t} \hat{\mathcal{E}}(\kappa, z_0) \kappa - \text{c.c.} \right) \right. \\ &\quad \times \left( \int d\kappa' e^{i(\kappa'\beta c - \omega)t} \hat{\mathcal{E}}(\kappa', z_0) \frac{\kappa' c - \beta\omega}{\kappa'\beta c - \omega} + \text{c.c.} \right) \\ &\quad + \int_{-\infty}^{\infty} dt \left( \int d\kappa e^{i(\kappa\beta c - \omega)t} \hat{\mathcal{E}}(\kappa, z_0) \left( \kappa^2 - \frac{\omega^2}{c^2} \right) + \text{c.c.} \right) \\ &\quad \times \left. \left( \int d\kappa' e^{i(\kappa'\beta c - \omega)t} \hat{\mathcal{E}}(\kappa', z_0) \frac{\kappa' c - \beta\omega}{(\kappa'\beta c - \omega)^2} - \text{c.c.} \right) \right\}.\end{aligned}\quad (16)$$

Integration over time introduces  $\delta$ -functions into the above integrals. It is easy to see that the first term in curly braces vanishes and the second term gives,

$$\begin{aligned}mc^2 \Delta\gamma_2 &= \\ &= \frac{ir_0^2 e^2 \pi}{2mc^2 \beta \gamma} \left( \int_{-\infty}^{\infty} d\kappa \hat{\mathcal{E}}(\kappa, z_0) \hat{\mathcal{E}}(\kappa', z_0) \left[ \kappa \frac{\kappa' - \beta\omega/c}{\kappa'\beta - \omega/c} \right. \right. \\ &\quad \left. \left. + \frac{(\kappa' - \beta\omega/c)(\kappa^2 - \omega^2/c^2)}{(\kappa'\beta - \omega/c)^2} \right] - \text{c.c.} \right).\end{aligned}\quad (17)$$

In the last equation  $\kappa' = -\kappa + 2\omega/(\beta c)$ . If we set  $\beta = 1$  in the last equation, then it can be simplified to

$$\Delta\gamma_2 = -\frac{i\omega r_0^2 e^2 \pi}{2m^2 c^5 \gamma} \left( \int_{-\infty}^{\infty} d\kappa \hat{\mathcal{E}}(\kappa, z_0) \hat{\mathcal{E}}(\kappa', z_0) - \text{c.c.} \right). \quad (18)$$

This equation shows that the energy gain in the second order is due to the interaction of the particle with two waves, with the wave numbers  $\kappa_1$  and  $\kappa_2$ , such that

$$\kappa_1 + \kappa_2 = \frac{2\omega}{\beta c}. \quad (19)$$

We can express  $\gamma_2$  in terms of the original field  $E_z(z)$  using the inverse Fourier transformation:  $\hat{\mathcal{E}}(\kappa, z_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{z0}(z) e^{-i\kappa(z-z_0)} dz$ . Substituting this equation into Eq. (18) gives

$$\Delta\gamma_2 = -\frac{i\omega r_0^2 e^2}{4m^2 c^5 \gamma} \left( \int_{-\infty}^{\infty} dz E_{z0}^2(z) e^{-2i(z-z_0)\omega/c} - \text{c.c.} \right). \quad (20)$$

Since  $\gamma_2$  is a second order effect it is proportional to the square of the electric field. It also scales as  $\gamma^{-1}$  with the energy, in contrast to the  $\gamma^{-2}$  dependence of the first order term Eq. (12). The phase factor  $e^{2iz_0\omega/c}$  in this integral shows that the second order energy gain depends on the relative phase of the RF field and reaches maximum for a specific choice of this phase.

One can also calculate the second order perturbation of the transverse velocity  $\Delta v_2 = v_{r2}(t = \infty)$  which describes a radial deflection of a particle passing through the RF cavity. Omitting the derivation we present here the final result:

$$\Delta v_2 = \frac{r_0 e^2}{4m^2 c^3 \gamma^2} \left( \int_{-\infty}^{\infty} dz E_{z0}^2(z) e^{-2i(z-z_0)\omega/c} + \text{c.c.} - 2 \int_{-\infty}^{\infty} dz |E_{z0}|^2(z) \right). \quad (21)$$

Note that  $\Delta v_2$  and  $\Delta\gamma_2$  are also related through the Panofsky-Wentzel relation (14). Eq. (21) agrees with the result of Ref. [11] derived for an infinitely long periodic structure.

## COMPUTER SIMULATIONS

To illustrate the nonlinear energy gain derived in the previous section we solved numerically the equations of motion Eq. (4) and Eq. (5) for a particle passing through an RF cavity. The electric field  $E_{z0}$  of a simulated RF cavity is assumed to be given by the following formula:

$$E_{z0}(z) = E_0 H(z) \frac{1}{2} (e^{i\kappa_1 z} + e^{i\kappa_2 z}), \quad (22)$$

with  $\kappa_1 = 0.5\omega/c\beta$  and  $\kappa_2 = 1.5\omega/c\beta$ . These values of  $\kappa_1$  and  $\kappa_2$  were chosen so that they satisfy the equation (19). The function  $H(z)$  is  $H =$

$(1/2) \left( \tanh \frac{z+L/2}{l} - \tanh \frac{z-L/2}{l} \right)$ —it defines the extent of the interaction region and imitates the finite length of the cavity. We used the following parameters for the simulation:  $L = 50$  cm,  $\omega = 2\pi \times 11.4$  GHz,  $\gamma = 10$ ,  $E_0 = 30$  MV/m,  $l = 0.4$  cm. Note that according to Eq. (1) the

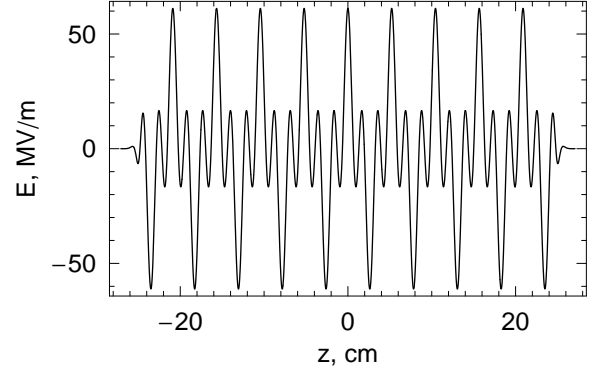


Figure 1: The plot of the electric field on the axis.

physical electric field on axis is  $2\text{Re}[E_{z0}(z)e^{-i\omega t}]$ ; the plot of the quantity  $2\text{Re}[E_{z0}(z)]$  is shown in Fig. 1.

The particles were injected into the cavity with an offset  $r_0$  parallel to the axis and tracked until they exit the cavity. The difference in the energy gain  $\Delta E = mc^2(\gamma_e(r_0) - \gamma_e(0))$ , where  $mc^2\gamma_e(r_0)$  is the exit energy as a function of the initial offset, is shown in Fig. 2 and demonstrates a parabolic dependence in accordance with the theoretical formula (20). Fig. 3 shows the dependence of the differ-

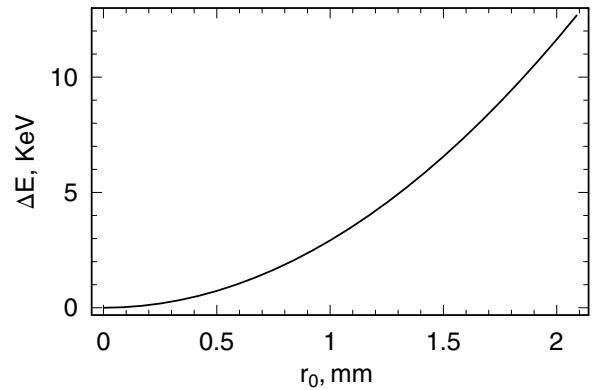


Figure 2: The dependence of the energy gain versus orbit offset.

ential energy gain  $\Delta E$  versus the peak accelerating electric field on axis  $E_0$ . For small values of  $E_0$  the result of the simulations agrees with the theoretical value Eq. (20), however, for  $E_0 \gtrsim 15$  MeV/m, the result of the simulation deviates from the theory. Note that our theory uses a perturbation approach and is valid in the limit of small accelerating field.

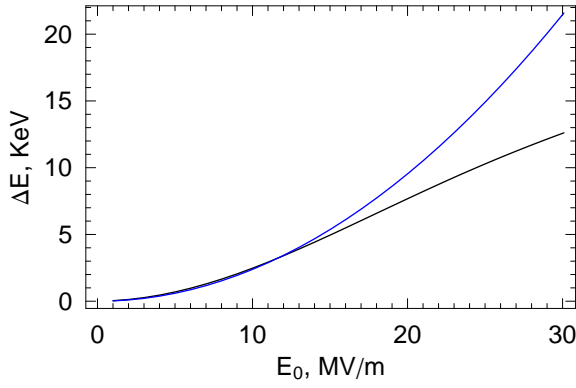


Figure 3: The dependence of the energy gain versus  $E_0$  (note that the peak accelerating field in the cavity is approximately equal to  $2E_0$ , see Fig. 1). The blue curve shows theoretical parabolic dependence, and the black one is the result of simulations.

## DISCUSSION

We demonstrated that the nonlinear interaction of an electron beam with the electromagnetic field generates an energy gain which is proportional to the square of the particle's offset—the dependence required for the beam conditioning. This differential energy gain decreases with  $\gamma$  (as  $\gamma^{-1}$ ). For not very large values of the beam energy,  $\gamma \sim 10$ , and accelerating field of the order of 30 MeV, it can generate the energy variation inside the beam of the order of 10–20 keV. Further increase of the accelerating field can put this values within the range of values promising for a modern x-ray SASE FEL application (for the LCLS, with account of beam compression in the linac, this value is of the order of 40-50 keV, [4]). However, the theory developed in this paper uses a perturbation approach and becomes invalid for large accelerating electric fields.

As mentioned above, in addition to the radial energy profile, such a field causes radial deflections of particles and

would contribute to the projected emittance growth. To some extent, this emittance growth can be suppressed if the particles' orbit are tilted and cross the axis at the center of the cavity. Indeed, as follows from Eq. (21), the transverse deflection is proportional to the offset of the particle and for a tilted orbit the kick from the first (before crossing the axis) and the last (after the crossing) part of the particle's trajectory will have different signs and would compensate each other. Note, though, that tilting the orbit makes that average  $r^2$  along the orbit smaller than for the parallel case and decreases the radial variation of the energy, see Eq. (18). The study of those effects for optimal values of the electric field in a realistic environment would require extensive numerical simulations.

## REFERENCES

- [1] *Linac Coherent Light Source (LCLS) Conceptual Design Report*, Report SLAC-R-593, SLAC (2002).
- [2] *TESLA: The Superconducting Electron Positron Linear Collider with an Integrated X-ray Laser Laboratory. Technical Design Report*, Report DESY-01-011, DESY (2001).
- [3] A. M. Sessler, D. H. Whittum, and L. Yu, *Phys. Rev. Lett.* **68**, 309 (1992).
- [4] P. Emma and G. V. Stupakov, *Phys. Rev. ST Accel. Beams* **6**, 030701 (2003).
- [5] N. A. Vinokurov, *Nucl. Inst. and Meth. A* **375**, 264 (1996).
- [6] A. Wolski, G. Penn, A. Sessler, and J. Wurtele, *Phys. Rev. ST Accel. Beams* **7**, 080701 (2004).
- [7] Emma P. and Stupakov G., in *Proceedings of 9th European Particle Accelerator Conference* (2004), p. 503.
- [8] C. B. Schroeder, E. Esarey, and W. P. Leemans, *Phys. Rev. Lett.* **93**, 194801 (2004).
- [9] A. A. Zholents, *Phys. Rev. ST Accel. Beams* **8**, 050701 (2005).
- [10] W. K. H. Panofsky and W. Wenzel, *Rev. Sci. Instr.* **27**, 967 (1956).
- [11] S. Hartman and J. Rosenzweig, *Phys. Rev.* **E47**, 2031 (1993).